

Observation of a Fast Evolution in a Parity-time-symmetric System

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In the parity-time-symmetric (PT-symmetric) Hamiltonian theory, the optimal evolution time can be reduced drastically and can even be zero. In this letter, we report our experimental simulation of the fast evolution of a PT-symmetric Hamiltonian in a nuclear magnetic resonance quantum system. The experimental results demonstrate that the PT-symmetric Hamiltonian system can indeed evolve much faster than the quantum system, and the evolution time can be arbitrarily close to zero.

Keywords: parity-time-symmetric Hamiltonian; non-Hermiticity; nuclear magnetic resonance; brachistochrone; duality quantum computation.

1. Introduction

The brachistochrone problem, i.e., the minimum time evolution between two states, is an interesting and important problem. In quantum mechanics, the brachistochrone between two states is bounded by the maximum difference of the eigenvalues of the Hamiltonian [1, 2, 3, 4, 5, 6, 7, 8]. Brachistochrone has important applications. For instance, the time-optimal approach to the quantum algorithmic complexity has attracted much interest recently [9, 10].

The Hermiticity requirement of a Hamiltonian guarantees that its eigenvalues are real. It also implies that the evolutionary operator $e^{-\frac{i}{\hbar}Ht}$ is unitary. However, Hermiticity is a sufficient condition but not necessary for real eigenvalues, and the entire spectrum of a wide class of non-Hermitian Hamiltonians

can also be real. Among these Hamiltonians [11] is a class that is PT symmetric. The PT-symmetric Hamiltonian has been investigated intensively in recent years, both in theory [11, 13, 14, 15, 16, 17, 18, 12, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31] and in experiments [32, 33, 34, 35, 36]. PT symmetries have been experimentally observed in table-top optical systems [32, 33, 34, 35] and in spin-polarized Rb atoms [36].

The novel character of PT-symmetric Hamiltonians brings about many new features and may lead to interesting applications. Faster than Hermitian quantum mechanics evolution is one of these important aspects [12]. In this article, we design and carry out an experiment that simulates the time evolution of a PT-symmetric Hamiltonian with a nuclear magnetic resonance (NMR) quantum system. We build a system in Hilbert space that admits both unitary and non-unitary evolution, and observe the time evolution of a PT-symmetric Hamiltonian. The experimental result shows that the minimal evolutionary time in a PT-symmetric system is faster than that in the Hermitian case, and can be arbitrarily close to zero.

2. Theoretical Frame

A simple PT-symmetric non-Hermitian Hamiltonian for a two-level system is

$$H = \begin{pmatrix} se^{i\alpha} & s \\ s & se^{-i\alpha} \end{pmatrix}. \quad (1)$$

According to Ref. [12], the largest and smallest eigenvalues are $E_{\pm} = 2s \cos \alpha$ and 0, respectively. The difference between them is

$$\omega = E_+ - E_- = 2s \cos \alpha. \quad (2)$$

Under the influence of $e^{-\frac{i}{\hbar}Ht}$, the PT-symmetric system that is initially in $|0\rangle = (1 \ 0)^T$ will go to [12]

$$e^{-\frac{i}{\hbar}Ht} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{e^{-\frac{i}{\hbar}ts \cos \alpha}}{\cos \alpha} \begin{pmatrix} \cos(\frac{\omega t}{2\hbar} - \alpha) \\ -i \sin(\frac{\omega t}{2\hbar}) \end{pmatrix}. \quad (3)$$

It takes the time

$$\tau = \frac{2\hbar}{\omega} \left(\alpha + \frac{\pi}{2} \right) \quad (4)$$

to evolve to state $|1\rangle = (0 \ 1)^T$. When $\alpha \rightarrow -\pi/2$, it approaches zero, which is an impossible task for a regular Hermitian Hamiltonian.

For comparison, the equivalent Hermitian Hamiltonian, H_0 , was calculated [12]

$$H_0 = \begin{pmatrix} s \cos \alpha & s \cos \alpha \\ s \cos \alpha & s \cos \alpha \end{pmatrix}, \quad (5)$$

the eigenvalues of which are $E_{\pm} = 2s \cos \alpha$ and 0, respectively. Here, $E_+ - E_- = \omega$, which is exactly the same as that in the PT-symmetric case.

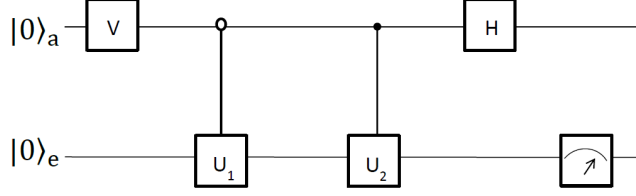


Figure 1: Quantum circuit for a PT-symmetric system. Operations from left to right are a single-qubit operation V , C_{0-U_1} (a 0-controlled U_1), C_{1-U_2} (a 1-controlled U_2), and a Hadamard operation H . When the ancillary qubit is in $|0\rangle_a$, the final state of work qubit is $e^{-\frac{i}{\hbar}Ht}|0\rangle_e$.

The evolution under this equivalent Hermitian Hamiltonian is given by

$$e^{-\frac{i}{\hbar}H_0t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = e^{-\frac{i}{\hbar}ts \cos \alpha} \begin{pmatrix} \cos\left(\frac{s \cos \alpha}{\hbar}t\right) \\ -i \sin\left(\frac{s \cos \alpha}{\hbar}t\right) \end{pmatrix}, \quad (6)$$

and the time it takes to evolve to the final state $|1\rangle$ is

$$\tau_0 = \frac{\pi \hbar}{\omega}, \quad (7)$$

which is constant for a fixed ω .

3. Construction of a PT-symmetric Hamiltonian System

We now construct a system with a PT-symmetric Hamiltonian Eq. (1) and simulate its time evolution. The vital part of the simulation is achieved using the idea of duality quantum computing [37, 38]. Duality quantum computing can be achieved by using an ancilla qubit using a conventional quantum computer [37, 38]. The principle to simulate a duality quantum computing is shown in a quantum circuit in Fig. 1. Simulating the non-unitary evolution of a PT-symmetric quantum system requires the determination of the explicit forms of the operators in Fig. 1.

For a conventional quantum computer, the idea to use an extended space consisted of a system and an ancilla for simulating a non-unitary evolution in a PT-symmetric system by a unitary evolution in an extended space was proposed in Ref. [39]. This is similar to the our scheme used in the experiment in this work.

The system we use contains two qubits: the work qubit e and the ancillary qubit a . A qubit is a two-level quantum system that is the building block in quantum information processing. The input 2-qubit state on the leftmost is $|0\rangle_a|0\rangle_e$. We then perform the V unitary operation,

$$V = \begin{pmatrix} \cos \phi_V & -\sin \phi_V \\ \sin \phi_V & \cos \phi_V \end{pmatrix}, \quad (8)$$

on the ancillary qubit, where ϕ_V is

$$\arccos \left(\frac{\sqrt{\left(2s \frac{\sin \frac{\omega t}{2\hbar}}{\omega}\right)^2 + \cos^2 \frac{\omega t}{2\hbar}}}{\sqrt{\left(2s \frac{\sin \frac{\omega t}{2\hbar}}{\omega}\right)^2 + \cos^2 \frac{\omega t}{2\hbar} + \left(2s \sin \alpha \frac{\sin \frac{\omega t}{2\hbar}}{\omega}\right)^2}} \right). \quad (9)$$

Next, we apply two controlled unitary operations,

$$C_{0-U_1} = \begin{pmatrix} U_1 & 0 \\ 0 & I \end{pmatrix}, \quad C_{1-U_2} = \begin{pmatrix} I & 0 \\ 0 & U_2 \end{pmatrix},$$

where

$$U_1 = \begin{pmatrix} \cos \phi_{U_1} & i \sin \phi_{U_1} \\ i \sin \phi_{U_1} & \cos \phi_{U_1} \end{pmatrix}, \quad (10)$$

$$\phi_{U_1} = \arcsin \left(-\frac{2s \frac{\sin \frac{\omega t}{2\hbar}}{\omega}}{\sqrt{\left(2s \frac{\sin \frac{\omega t}{2\hbar}}{\omega}\right)^2 + \cos^2 \frac{\omega t}{2\hbar}}} \right), \quad (11)$$

and $U_2 = \sigma_z$. Finally, a Hadamard operation is utilized on the work qubit e . Here, t is the evolution time in the PT-symmetric system, s is a parameter in the Hamiltonian in Eq. (1), and ω is the difference between the two eigenvalues of the Hamiltonian.

After performing operations shown in Fig. 1, the final state becomes

$$\frac{q}{\sqrt{2}} \left[|0\rangle_a e^{-\frac{i}{\hbar} H t} |0\rangle_e + |1\rangle_a \frac{1}{q} (\cos \phi_V U_1 - \sin \phi_V U_2) |0\rangle_e \right], \quad (12)$$

where q is

$$e^{\frac{i}{\hbar} t s \cos \alpha} / \sqrt{\left(2s \frac{\sin \frac{\omega t}{2\hbar}}{\omega}\right)^2 + \cos^2 \frac{\omega t}{2\hbar} + \left(2s \sin \alpha \frac{\sin \frac{\omega t}{2\hbar}}{\omega}\right)^2}, \quad (13)$$

which is a non-zero number and tends to $1/\sqrt{3}$ as $\alpha \rightarrow -\pi/2$ and $t = \tau$. If we observe the work qubit conditioned on the ancillary qubit to be in state $|0\rangle_a$, the evolution associated with the work qubit becomes

$$e^{-\frac{i}{\hbar} H t} |0\rangle_e, \quad (14)$$

which is the PT-symmetric Hamiltonian evolution.

It is worth explaining the symbols. We use t to denote the evolution time in the PT-symmetric quantum system. The time it takes to complete the evolution in the work-ancilla two qubits system is designated as \tilde{t} . The time it takes for the PT-symmetric quantum system to evolve from $|0\rangle$ to $|1\rangle$ is indicated as τ and the corresponding time it takes to complete the operations in the work-ancilla two qubits system represented by $\tilde{\tau}$. As we construct the PT-symmetric

system with a one-qubit-subspace of a two-qubit-Hilbert space, in which the sub-system evolves non-unitarily while the whole system evolves unitarily, the evolving time \tilde{t} of the whole system depends on the evolution time t of the PT-symmetric system, and vice versa. The evolving time τ for the PT-symmetric system can approach zero when α tending to $-\frac{\pi}{2}$, which is faster than the counterpart Hermitian system.

4. Experimental Realization

We simulated the evolution process in an NMR quantum system, C^{13} -labeled chloroform that consists of two qubits. The C^{13} nuclear spin works as the work qubit and the proton nuclear spin works as the ancillary qubit. We begin from the state $|0\rangle_a|0\rangle_e$. We then evolve the PT-symmetric system to some time t by applying the corresponding operations given in Fig. 1. Next, we measure the state of the work qubit conditioned on the ancillary qubit a being in state $|0\rangle_a$. By varying the instant t , we observe the state evolution of the PT-symmetric system.

The following NMR notations are adopted. The free evolution of the two-qubit system for a period of X is denoted as $[X]$:

$$[X] = e^{-i\frac{\pi J X}{2}\sigma_z^a\sigma_z^e}, \quad (15)$$

where $J = 215.23$ Hz is the coupling constant between C^{13} and H^1 . A rotation of spin m through angle ϕ about axis j is denoted as $[\phi]_j^m$, and $[\phi]_j^m = e^{-i\phi\sigma_j^m/2}$.

The spatial-averaging method [40] was used to prepare the pseudo-pure state $|0\rangle_a|0\rangle_e$. The single-qubit rotation operation V is realized by a pulse

$$[2\phi_V]_y^a \quad (16)$$

on the ancillary qubit. Here, C_{0-U_1} and C_{1-U_2} are realized by the following two pulse sequences:

$$\begin{aligned} [\frac{\pi}{2}]_y^e &\rightarrow [\frac{\phi_{U_1}}{2\pi J}] \rightarrow [\pi]_x^{a,e} \rightarrow [\frac{\phi_{U_1}}{2\pi J}] \\ &\rightarrow [\pi]_{-x}^{a,e} \rightarrow [\frac{\pi}{2}]_{-y}^e \rightarrow [\phi_{U_1}]_{-x}^e, \end{aligned} \quad (17)$$

and

$$\begin{aligned} [\pi]_y^e &\rightarrow [\frac{1}{4J}] \rightarrow [\pi]_x^{a,e} \rightarrow [\frac{1}{4J}] \rightarrow [\pi]_{-x}^{a,e} \rightarrow [\frac{\pi}{2}]_{-y}^e \\ &\rightarrow [\frac{\pi}{2}]_x^e \rightarrow [\frac{\pi}{2}]_{-y}^e \rightarrow [\frac{\pi}{2}]_y^a \rightarrow [\frac{\pi}{2}]_x^a \rightarrow [\frac{\pi}{2}]_{-y}^a, \end{aligned} \quad (18)$$

respectively. Finally, the pulse sequence

$$[\frac{\pi}{2}]_y^a \rightarrow [\pi]_{-x}^a, \quad (19)$$

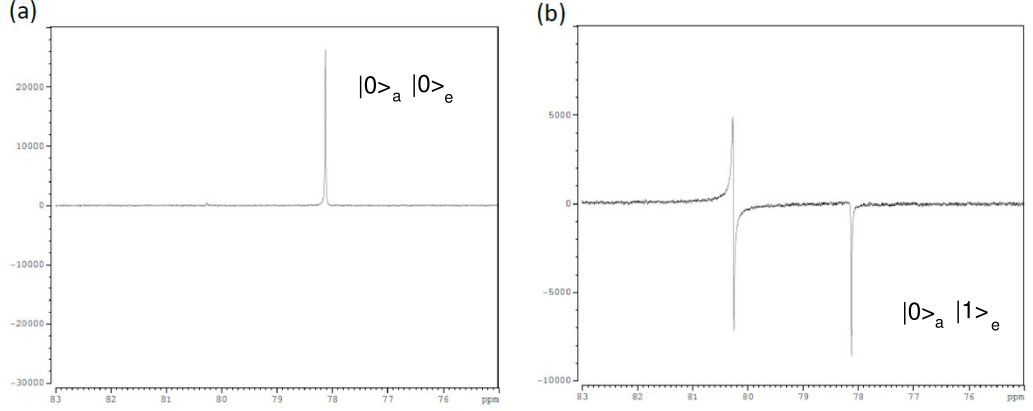


Figure 2: Typical spectra of the work qubit with $\alpha = -\pi/8$: (a) pseudo-pure state at the beginning of evolution; (b) final state after evolving for a time of τ .

is applied to implement the Hadamard operation on the ancillary qubit. The evolution is observed by looking at both the state of the ancillary and the work qubit. If the ancillary qubit is in state $|0\rangle_a$, the state of the work qubit gives the evolution under the PT-symmetric Hamiltonian.

Because we are interested in the behavior of the system near $\alpha = -\pi/2$, we restricted the parameter in the range of $\alpha \in (-\frac{\pi}{2}, 0]$. We performed a series of experiments with values of α at 0 , $-\frac{\pi}{4}$, $-\frac{3\pi}{8}$, $-\frac{7\pi}{16}$, $-\frac{15\pi}{32}$, and $-\frac{31\pi}{64}$. Fig. 2(a) shows the spectrum of the work qubit in the pseudo-pure state. A right single upward peak in the spectrum indicates that the work qubit and the ancillary qubit are all in state $|0\rangle$. Fig. 2(b) shows the spectrum of the work qubit for $\alpha = -\frac{\pi}{8}$. The downward peak on the right indicates that the two-qubit state is $|0\rangle_a|1\rangle_e$, whereas the peak on the left corresponds to the ancillary qubit in state $|1\rangle_a$, which is not of interest to the present study.

We now examine the simulation when $t = 0$. Here, V operation becomes an identity operator and it does not require time to simulate. The two free evolutions $[\phi_{U1}/(2\pi J)]$ pulses and the last single qubit pulse $[-\phi_{U1}]_x^e$ in the C_{0-U_1} in Eq. (17) also do not require any time. The other pulses in C_{0-U_1} and the whole pulse sequence of C_{1-U_2} and H still require a constant (with respect to t) time to complete. Thus, the simulation performed in the two-qubit system still requires time \tilde{t} to complete, even though t is zero.

Quantitative results for the evolution in the PT-symmetric system were obtained by performing quantum state tomography. Fig. 3 gives the density matrices of work qubit for $\alpha = -0.4844\pi$ at the beginning ($t = 0$), middle ($t = \tau/2$), and end ($t = \tau$) of the evolution. Fig. 3(a) is the density matrix at the beginning where the state is $|0\rangle$; Fig. 3(b) shows the state in the middle of the process; and Fig. 3(c) gives the final state. For comparison, we drew

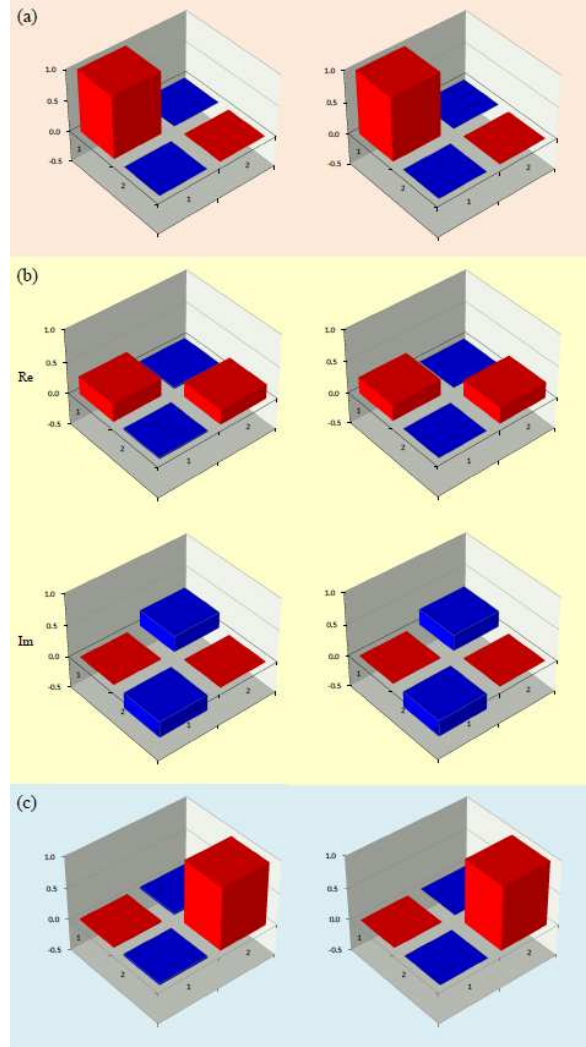


Figure 3: State tomography of the work qubit with $\alpha = -31\pi/64 = -0.4844\pi$: (a) pseudo-pure state $|0\rangle_e$, (b) middle state at $t = \tau/2$, and (c) final state. In each picture, the left side provides the experimental results, whereas the right side provides the theoretical results. (Online version in colour.)

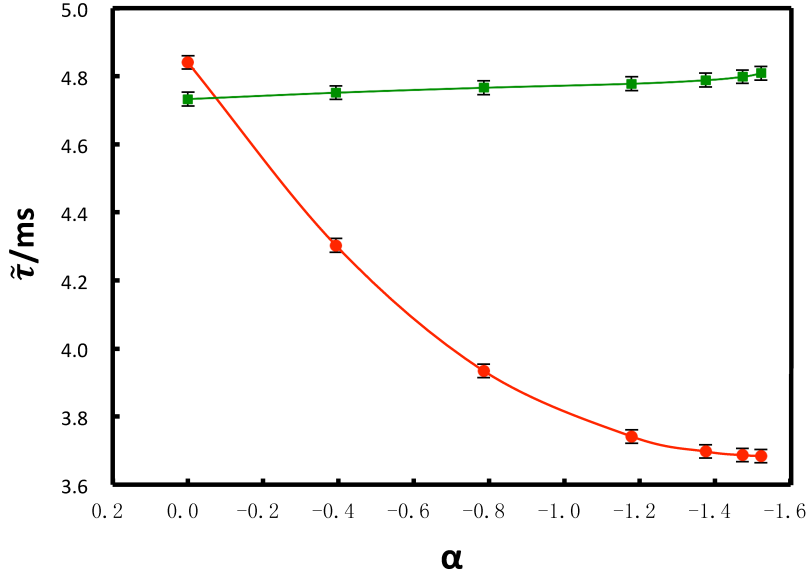


Figure 4: Total experimental time $\tilde{\tau}$ for simulating evolution from $|0\rangle$ to $|1\rangle$. The red circles denote the PT-symmetric case, while the green squares indicate the Hermitian case. The connected lines are used as visual guides only. (Online version in colour.)

the corresponding theoretical density matrices on the right side of each picture. Clearly, the experiments agree with theory very well.

The total experimental time $\tilde{\tau}$ to finish the simulation of the evolution from $|0\rangle_e$ to $|1\rangle_e$ in the two-qubit system is shown in Fig. 4. As α approaches $-\pi/2$, $\tilde{\tau}$ decreases remarkably; however, it does not reach zero. As the PT-symmetric Hamiltonian is realized in a larger quantum system, $\tilde{\tau}$ depends, not only on the evolution time τ in the PT-symmetric system, but also on the time it takes to set up the PT-symmetric system. The relation between t in the PT-symmetric system and \tilde{t} in the two-qubit system is determined through the four operations shown in Fig. 1. Of the four operations, V and C_{0-U1} are dependent on t , whereas C_{1-U2} and H are constant operations that are independent of t . In the simulation, t represents a parameter in determining the operations of V and C_{0-U1} .

The relation between τ and α predicted in Ref. [12] appears in the data remarkably well. An evolution faster than the Hermitian Hamiltonian system evolution is clearly observed. The evolution time τ taken by the PT-symmetric system to go from $|0\rangle$ to state $|1\rangle$ is shown in Fig. 5. In Fig. 5, the evolutionary time for the PT-symmetric Hamiltonian system approaches zero when α approaches $-\pi/2$.

For comparison, we simulated the evolution of the equivalent Hermitian Hamiltonian H_0 . The quantum circuit is similar to that for the PT-symmetric

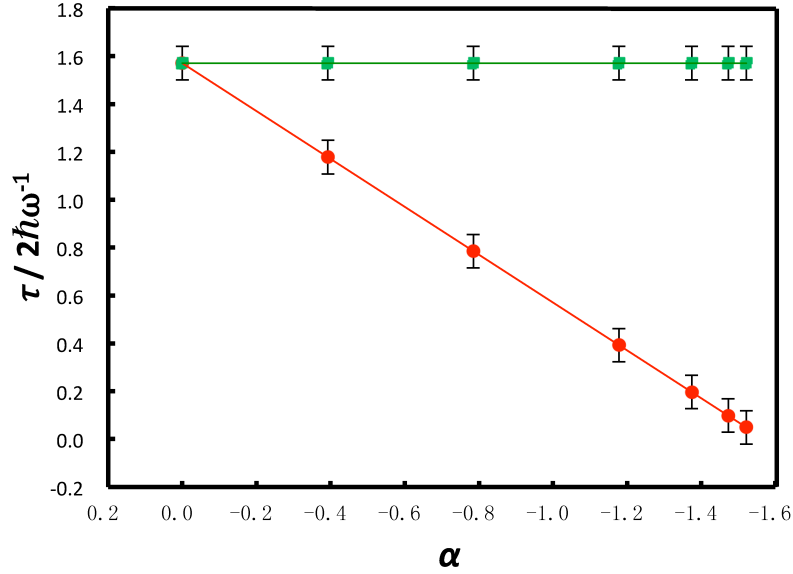


Figure 5: Evolution time τ versus α . The red circles indicate the PT-symmetric case, while the green squares denote the Hermitian case. (Online version in colour.)

case shown in Fig. 1. However, we substituted U_1 , U_2 and H by

$$\tilde{U}_1 = \tilde{U}_2 = \begin{pmatrix} \cos\left(\frac{s \cos \alpha t}{\hbar}\right) & -i \sin\left(\frac{s \cos \alpha t}{\hbar}\right) \\ -i \sin\left(\frac{s \cos \alpha t}{\hbar}\right) & \cos\left(\frac{s \cos \alpha t}{\hbar}\right) \end{pmatrix} \quad (20)$$

and V^\dagger , respectively, where V^\dagger is the Hermitian conjugate of the V operator in Eq. (8). The total time $\tilde{\tau}$ to implement the simulation and the evolution time τ for the equivalent Hermitian system are obtained and shown in Figs. 4 and 5, respectively. The evolutionary time τ for the Hermitian case is clearly constant regardless of the value of α . The faster-than-Hermitian evolution of the PT-symmetric system is evident.

5. Conclusions

In conclusion, we experimentally simulated the evolution of a PT-symmetric system in an NMR quantum system with two qubits. The faster than Hermitian quantum mechanics evolution of a PT-symmetric system predicted in Ref. [12] is clearly observed. When the parameter α approaches $-\pi/2$, the evolution time also approaches zero.

When the difference between the large and small eigenvalues of a Hermitian two-level quantum system is fixed, the fastest evolving time is invariant for a spin flipping in Hermitian quantum mechanics. However, for PT-symmetric quantum system, the brachistochrone time can be varied by changing the parameters in

the Hamiltonian. It can not only accelerate the evolution as predicted in Ref. [12] and demonstrated in this work, but also decelerate the evolving time as shown in Ref. [41].

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